# Radius of Convergence for gLV Systems

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We will use the Indirect Method of Lyapunov (citation) to obtain a radius of convergence (ROC) for a given equilibrium of the generalized Lotka Volterra system (gLV). Note, this is a systematic method but it does not in general obtain the maximum ROC.

## 1 The Natural System

The gLV system of size $n$ is defined as

$$\dot{x_i} = x_i \left( r_i + \sum_{j=1}^{n} \alpha_{ij} x_j \right) \qquad i \in [1, n] \tag{1.1}$$

such that $r_i$ is the intrinsic growth rate of population $x_i$ and $\alpha_{ij}$ is the "interaction from $x_j$ to $x_i$", the effect that $x_j$ has on the growth of $x_i$. The system can be summarized,

$$\dot{x} = x \circ (r + Ax) \qquad x, r \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n} \tag{1.2}$$

with $\circ$ defined as the Hadamard or element wise product (citation). This nonlinear system can be rearranged to partition the linear and nonlinear factors

$$\dot{x} = Jx + g(x)$$

for $J := Df(x) \mid_{x=0}$, the Jacobian evaluated at the origin, and $g(x) :=$ the remaining nonlinearities. Thus,

$$J = \begin{vmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & r_n \end{vmatrix} \quad \text{and,} \quad g(x) = (A \circ X_r)\, x \quad \text{for} \quad X_r := \begin{vmatrix} x_1 & \cdots & x_1 \\ \vdots & & \vdots \\ x_n & \cdots & x_n \end{vmatrix} \tag{1.3}$$

## 2 Lyapunov Analysis of the Natural System

We use the standard, quadratic Lyapunov function to interrogate the stability of the origin. However, this case is irrelevant itself because the trivial equilibrium of a gLV system is never stable (note, the eigenvalues of $J$ are the growth rates $r$ which are positive by definition), rather, it is derived to understand the translated system.

Let $V$ be the standard, quadratic Lyapunov function,

$$V(x) := x^T P x \quad \text{where } P \text{ solves} \quad J^T P + PJ = -I \tag{2.1}$$

Truly, there will be no positive, symmetric matrix P which can solve this because of the nature of J; it is only when we shift the equilibrium to a nontrivial fixed point does it become possible. Pretending for now,

$$\dot{V}(x) = -x^T x + 2x^T P g(x) = -x^T(1 - 2P(A \circ X_r))x$$

and

$$\dot{V}(x) \leq -(1 - 2\,\|P\|_{i,2}\,\|A \circ X_r\|_{i,2})\,\|x\|_2^2 \tag{2.2}$$

Well,

$$\|A \circ X_r\|_{i,2} \leq \|A \circ X_r\|_F \leq \|A\|_F\,\|X_r\|_F = \|A\|_F\,\sqrt{n}\,\|x\|_2$$

Therefore, we can conclude the local region where $-\dot{V}(x)$ is positive definite,

$$\|x\|_2 < r := \frac{1}{2\sqrt{n}\,\|P\|_{i,2}\,\|A\|_F} \implies 1 - 2\,\|P\|_{i,2}\,\|A \circ X_r\|_{i,2} > 0 \implies -\dot{V}(x) \text{ LPDF}$$

and if,

$$x \in \Omega : \{\bar{x} \mid V(\bar{x}) < \lambda_{min}(P)r^2\} \tag{2.3}$$

then,

$$\lambda_{min}(P)\,\|x\|^2 < V(x) = x^T P x \leq \lambda_{min}(P)r^2 \implies \|x\| < r$$

guarantees that the trajectories will remain within a region where $-\dot{V}(x)$ is LPDF. Ofcourse, the Lyapunov Theorem for Time Invariant System dictates that this is the region where trajectories will converge to the origin asymptotically.

# 3 Translation to Equilibria of Interest

Let $f = (f_1 ... f_n)$ be a stable (non-trivial) equilibrium of the natural system for which we desire to know the radius of convergence. We can use the following change of variables to derive a system with this equilibrium at the origin,

$$z = x - f \iff z + f = x \implies \dot{z} = \dot{x} \implies \dot{z} = (z + f) \circ (r + A(z + f)) \tag{3.1}$$

which can first be simplified using the properties of the Hadamard operator,

$$\dot{z} = z \circ (r + Az) + f \circ (r + Az) + (z + f) \circ (Af)$$

Recall, the nontrivial equilibrium $f = -A^{-1}(r) \implies Af = -r$, allowing,

$$\dot{z} = z \circ (r + Az) + f \circ (r + Az) + z \circ (-r) + f \circ (-r)$$

$$\dot{z} = (f \circ A)z + (z \circ A)z = J'z + g(z) \tag{3.2}$$

The eigenvalues of $J'$ are potentially negative, thus, we know that there exists a $P_z$ such that,

$$J'^T P_z + P_z J' = -I \quad \text{holds for} \quad V(z) = z^T P_z z,$$

therefore, our Lyapunov analysis will hold for this system which has the same nonlinear piece $g(z)$.

Hence, the region of convergence for this equilibrium is given by the set,

$$z \in \Omega_z : \{\bar{z} \mid V(\bar{z}) < \lambda_{min}(P_z)r_z^2\} \text{ for } r_z = \frac{1}{2\sqrt{n}\,\|P_z\|_{i,2}\,\|A\|_F} \tag{3.3}$$